Finiteness of relative equilibria in the planar generalized N-body problem with fixed subconfigurations

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MEGA,
Trento, June 2015

Joint work with Marshall Hampton
Central configurations in celestial mechanics

Consider $N$ bodies in space with masses $m_1, \ldots, m_N$ satisfying Newton’s laws of motion. A configuration $(p_1, \ldots, p_n)$ is central if it collapses to a point by scaling if started with velocity zero.

Example

Let $c$ denote the center of mass and $r_{ij}$ the relative distances.

Central $\iff \exists \lambda \in \mathbb{R}_{>0} : \forall j \in \{1, \ldots, n\} : \lambda (c - p_j) = \sum_{i \neq j} \frac{m_i (p_i - p_j)}{r_{ij}^3}$
How many central configurations are there?

(Up to translation, scaling and rotation)

**Theorem (Hampton and Moeckel (2006))**

*For any choice of 4 positive masses there are only finitely many 2-dimensional central configurations.*

**Theorem (Hampton, Jensen, 2011)**

*Except for some explicitly given exceptional cases for the masses, the Newtonian 5-body problem has only finitely many 3-dimensional central configurations.*

**Theorem (Albouy, Kaloshin, 2012)**

*Except for some explicitly given exceptional cases for the masses, the Newtonian 5-body problem has only finitely many 2-dimensional central configurations.*
How many central configurations are there?

There are other results in this direction for general $N$, BUT:

Theorem (Roberts (1999))

If negative masses were allowed, then it would be possible to have a continuum of central configurations.

From now on: We consider the planar case.

Smale’s 6th problem: Given $N$ point masses, are there only finitely many ways to form central configurations?

(Up to translation, scaling and rotation)
Our theorem — or rather Lindstrom 2001

Theorem (Lindstrom, 2001)

For each fixed configuration of (N-1)-bodies, there are at most finitely many ways to place an additionally given body.*

Example

Where could we place a point mass with mass equal to 2?

Our contribution:

▶ Using tropical geometry we give an easy proof.
▶ We generalise this to any integral exponent $D \geq 2$.

*If total mass is non-zero, (N-1) masses are non-zero and mass of last body is given.
The polynomial system

Variables:
- pairwise distances $r_{1N}, \ldots, r_{(N-1)N}$

Parameters:
- masses $m_1, \ldots, m_N$
- pairwise distances $r_{12}, \ldots, r_{1(N-1)}, r_{23}, \ldots, r_{(N-2)(N-3)}$

If masses sum to non-zero:
A (normalised) central configuration must satisfy equations:
- Asymmetric “Albouy-Chenciner” equations

$$AC_{ij} = \sum_{k=1}^{n} m_k (r_{ik}^{-3} + 1)(r_{jk}^2 - r_{ik}^2 - r_{ij}^2) = 0$$

- Cayley-Menger determinants $=0$
Cayley-Menger equations

Heron's formula

\[
\text{Area} = \sqrt{s(s - a)(s - b)(s - c)} = \frac{1}{4} \sqrt{-\det \begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & a^2 & b^2 \\
1 & a^2 & 0 & c^2 \\
1 & b^2 & c^2 & 0
\end{pmatrix}}
\]

We consider 2-dimensional configurations
⇒ 3-dimensional volume is zero.

\[
CM_{ijkN} := \det \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & r_{ij}^2 & r_{ik}^2 & r_{iN}^2 \\
1 & r_{ij}^2 & 0 & r_{jN}^2 & r_{jN}^2 \\
1 & r_{ik}^2 & r_{jk}^2 & 0 & r_{kN}^2 \\
1 & r_{iN}^2 & r_{jN}^2 & r_{kN}^2 & 0
\end{pmatrix} = 0
\]

Newton polytope is the simplex \(\text{conv}(4e_i, 4e_j, 4e_k, 0) \subseteq \mathbb{R}^{N-1}\)
Initial forms and initial ideals

Consider the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. Let $\omega \in \mathbb{R}^n$.

- The $\omega$-degree of a monomial $x_1^{a_1} \cdots x_n^{a_n}$ with $a \in \mathbb{N}^n$ is $\langle \omega, a \rangle$.
- The initial form $\text{in}_\omega(f)$ of a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is the sum of terms with maximal $\omega$-degree.

Example:

$$\text{in}_{(1,2)}(x_1^4 + 2x_2^2 + x_1x_2 + 1) = x_1^4 + 2x_2^2$$

- The initial ideal of an ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is defined as

$$\text{in}_\omega(I) = \langle \text{in}_\omega(f) \rangle_{f \in I}$$
Tropical Varieties

Definition (Speyer, Sturmfels, 2004)
If \( I \subseteq \mathbb{C}[x_1, \ldots, x_n] \) is an ideal then we define

\[
T(I) := \{ \omega \in \mathbb{R}^n : \text{in}_\omega(I) \text{ is monomial-free} \}.
\]

Theorem (Bieri-Groves)

Let \( d = \dim(V(I)) \) where \( V(I) \subseteq (\mathbb{C}^*)^n \).

The tropical variety \( T(I) \) has dimension \( d \).
Example

The tropical variety of a principal ideal is called a tropical hypersurface. $T(\langle x_1 + x_2 + x_3 \rangle) \subseteq \mathbb{R}^3$ is the union of three 2-dimensional cones:

Theorem

Any tropical variety is an intersection of hypersurfaces:

$$T(I) = \bigcap_{f \in I} T(\langle f \rangle)$$
Proof strategy — Inspired by Gfan computations

Let

- $\mathcal{A} := \{AC_{N1}, AC_{N2}, \ldots, AC_{N(N-1)}\}$
- $\mathcal{C} := \{CM_{i,j,k,N}| 1 \leq i < j < k < N\}$
- $I := \langle \mathcal{A}, \mathcal{C} \rangle$

\[
|V(I) \cap (\mathbb{R}^+)^{N-1}| < \infty
\]
\[
\iff |V(I) \cap (\mathbb{C}^*)^{N-1}| < \infty
\]
\[
\iff \text{dim}(V(I) \cap (\mathbb{C}^*)^{N-1}) = 0
\]
\[
\iff T(I) = \{0\}
\]
\[
\iff \bigcap_{f \in I} T(\langle f \rangle) = \{0\}
\]
\[
\iff T(\langle \mathcal{A} \rangle) \cap \bigcap_{f \in \mathcal{C}} T(\langle f \rangle) = \{0\}
\]
Proof strategy

Lemma
The tropical prevariety $\bigcap_{C \in \mathcal{C}} T(C) \subseteq \mathbb{R}^{N-1}$ defined by $\mathcal{C}$ is the 2-skeleton of the normal fan of the $(N-1)$-dimensional standard simplex in $\mathbb{R}^{N-1}$.

Matroid argument?
Proof strategy

Lemma
\[ T(\langle \mathcal{A} \rangle) \cap \{ \omega \in \mathbb{R}^{N-1} : \sum_i \omega_i \leq 0 \} \cap \bigcap_{C \in \mathcal{C}} T(C) = \{0\} \]

- For each 2-dimensional cone in \( \bigcap_{C \in \mathcal{C}} T(C) \subseteq \mathbb{R}^{N-1} \) find an initial form which is a monomial.
- In the case where \( D = 2 \), this is not possible — we must cancel terms.
- Argument independent of parameters!

Now
\[ T(I) \subseteq \{ \omega \in \mathbb{R}^{N-1} : \sum_i \omega_i > 0 \} \cup \{0\} \]

It is a tropical variety \( \Rightarrow \) Must be balanced \( \Rightarrow \) \( T(I) = \{0\} \).
Q.E.D.
References

- Lindstrom: “The number of planar central configurations is finite when $N-1$ mass positions are fixed” (2001)
- Hampton, Jensen: “Finiteness of spatial central configurations in the five-body problem” (2011)
- Albouy, Kaloshin: “Finiteness of central configurations of five bodies in the plane” (2012)

...The list goes back to Euler

- Bieri, Groves: “The geometry of the set of characters induced by valuations” (1984)